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On the existence of polynomial first integrals of quadratic homogeneous systems of ordinary differential equations

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Abstract

We consider systems of ordinary differential equations with a quadratic homogeneous right-hand side. We give a new simple proof of an earlier result, which gives the necessary conditions for the existence of polynomial first integrals. The necessary conditions for the existence of a polynomial symmetry field are given. It is proved that an arbitrary homogeneous first integral of a given degree is a linear combination of a fixed set of polynomials.

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1. Introduction

In this paper we study the system of ordinary differential equations with a quadratic homogeneous right-hand side:

$$\dot{x}_i = f_i(x_1, \ldots, x_n) \qquad f_i = \sum_{j,k=1}^{n} a_{ijk} x_j x_k \qquad a_{ijk} \in \mathbb{C} \qquad i = 1, \ldots, n. \tag{1.1}$$

Systems of such a form arise in many problems of classical mechanics: Euler–Poincaré equations on Lie algebras, the Lotka–Volterra systems, etc.

The main concern of this paper is to find the values of the parameters a_{ijk} for which equations (1.1) have first integrals.

In paper [2] the necessary conditions are found for the existence of polynomial first integrals of the system

$$\dot{x}_i = V_i(x_1, \ldots, x_n) \qquad i = 1, \ldots, n \tag{1.2}$$

where $V_i \in \mathbb{C}[x_1, \ldots, x_n]$ are homogeneous polynomials of weighted degree $s \in \mathbb{N}$. In the case $s = 2$ we obtain equations (1.1).

The method given in [2] is based on ideas of Darboux [1, 6, 7] who used a special type of particular solutions of the system (1.2)

$$x_i(t) = d_i \phi(t) \qquad i = 1, \ldots, n$$

where $\phi(t)$ satisfies the differential equation $\dot{\phi} = \lambda\phi^s$, λ is an arbitrary number and $d = (d_1, \ldots, d_n)^T \neq 0$ is a solution of the following algebraic system:

$$V_i(d) = \lambda d_i \qquad i = 1, \ldots, n.$$

In this paper we generalize this method.

It was shown in [8, 10] that the weighted degree of a polynomial first integral of the system (1.1) is a certain integer linear combination of Kovalevskaya exponents (see [9]). In section 2 we give a new simple proof of this result. In section 3 a similar theorem for polynomial symmetry fields is proven. As an example, we consider the well known Halphen equations. Section 4 contains our main result. We present so-called *base functions* and prove that every homogeneous polynomial first integral of a fixed degree is a certain linear combination of the corresponding base functions. In section 5 we give an application of previous results to planar homogeneous quadratic systems where necessary and sufficient conditions for the existence of polynomial first integrals in terms of Kovalevskaya exponents are found.

2. The existence of the polynomial first integral. Necessary conditions

Following paper [2], we consider the solution $C = (c_1, \ldots, c_n)^T \neq (0, \ldots, 0)^T$ of algebraic equations

$$f_i(c_1, \ldots, c_n) + c_i = 0 \qquad i = 1, \ldots, n. \tag{2.1}$$

We define the *Kovalevskaya matrix* K [3]

$$K_{ij} = \frac{\partial f_i}{\partial x_j}(C) + \delta_{ij} \qquad i, j = 1, \ldots, n.$$

where δ_{ij} is the Kronecker symbol. Let us assume that K can be transformed to diagonal form

$$K = \mathrm{diag}(\rho_1, \ldots, \rho_n).$$

The eigenvalues $\rho_1, \ldots, \rho_n$ are called *Kovalevskaya exponents*.

Lemma 1 [3]. *Vector C is an eigenvector of the matrix K with eigenvalue $\rho_1 = -1$.*

Consider the following linear differential operators:

$$\begin{aligned} D_+ &= \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} \qquad & D_0 &= \sum_{i,j=1}^n K_{ij} x_j \frac{\partial}{\partial x_i} \\ U &= \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \qquad & D_- &= \sum_{i=1}^n c_i \frac{\partial}{\partial x_i} \end{aligned} \tag{2.2}$$

which satisfy relations

$$[D_-, D_+] = D_0 - U \qquad [D_0, D_-] = D_- \tag{2.3}$$

where $[A, B] = AB - BA$.

Theorem 2 [8, 10]. *Suppose that the system (1.1) possesses a homogeneous polynomial first integral F_M of degree M, and $\rho_1 = -1, \rho_2, \ldots, \rho_n$ are Kovalevskaya exponents. Then there exists a set of non-negative integers $k_2, \ldots, k_n$ such that*

$$k_2\rho_2 + \cdots + k_n\rho_n = M \qquad k_2 + \cdots + k_n \leqslant M. \tag{2.4}$$

Proof. By definition of a first integral $D_+F_M = 0$. Considering identities

$$D_-^l(D_+F_M) = 0 \qquad \text{for} \quad l \in \mathbb{N} \tag{2.5}$$

we obtain the following set of polynomials:

$$F_M, F_{M-1}, \ldots, F_{\rho+1}, F_\rho$$

defined by the recursive relations

$$D_-F_{i+1} = (M-i)F_i \qquad i = \rho, \ldots, M-1$$

where the number $1 \leqslant \rho \leqslant M$ is determined by the condition $D_-F_\rho = 0$. Using (2.3) and (2.5) we deduce the following chain of equations:

$$\begin{aligned} &D_0F_M = MF_M - D_+F_{M-1} \\ &D_0F_{M-1} = MF_{M-1} - D_+F_{M-2} \\ &\cdots \\ &D_0F_{\rho+1} = MF_{\rho+1} - D_+F_\rho \\ &D_0F_\rho = MF_\rho. \end{aligned} \tag{2.6}$$

Let $J_1, \ldots, J_n$ be linearly independent eigenvectors of the Kovalevskaya matrix K corresponding to the eigenvalues $\rho_1 = -1, \rho_2, \ldots, \rho_n$. According to lemma 1 we can always put $J_1 = C$.

We now consider the linear change of variables

$$x_i = \sum_{j=1}^{n} L_{ij} y_j \qquad i = 1, \ldots, n \tag{2.7}$$

where $L = (L_{ij})$ is a non-singular matrix defined by

$$L = (C, J_2, \ldots, J_n),$$

then obviously

$$L^{-1}KL = \mathrm{diag}(-1, \rho_2, \ldots, \rho_n).$$

With the help of (2.7) and lemma 1 one finds the following expressions for the operators D_0, D_- in the new variables:

$$D_0 = \sum_{i=1}^{n} \rho_i y_i \frac{\partial}{\partial y_i} \qquad D_- = -\frac{\partial}{\partial y_1}$$

and equation (2.6) becomes

$$\left(\rho_2 y_2 \frac{\partial}{\partial y_2} + \cdots + \rho_n y_n \frac{\partial}{\partial y_n}\right) F_\rho = MF_\rho. \tag{2.8}$$

We can write the polynomial F_ρ as follows:

$$F_\rho = \sum_{|k|=\rho} A_{k_2,\ldots,k_n} y_2^{k_2}, \ldots, y_n^{k_n} \qquad |k| = k_2 + \cdots + k_n \qquad k_i \in \mathbb{Z}_+. \tag{2.9}$$

Substituting (2.9) into (2.8), one obtains the following linear system:

$$(k_2\rho_2 + \cdots + k_n\rho_n)A_{k_2,\ldots,k_n} = MA_{k_2,\ldots,k_n} \qquad \text{for} \quad |k| = \rho. \tag{2.10}$$

Taking into account that F_ρ is not zero identically, we conclude that there exists at least one non-zero set $k_2, \ldots, k_n$, $|k| \leqslant M$ such that

$$k_2\rho_2 + \cdots + k_n\rho_n = M. \tag{2.11}$$

This relation implies (2.4). □

Remark. Theorem 2 does not impose any restrictions on $\mathrm{grad}(F_M)$ calculated at the point C. Thus, it generalizes the theorem of Yoshida [3, p 572], who used essentially the condition $\mathrm{grad}(F_M) \neq 0$.

Corollary 1. *The Halphen equations*

$$\begin{aligned} \dot{x}_1 &= x_3x_2 - x_1x_3 - x_1x_2 \\ \dot{x}_2 &= x_1x_3 - x_2x_1 - x_2x_3 \\ \dot{x}_3 &= x_2x_1 - x_3x_2 - x_3x_1 \end{aligned} \tag{2.12}$$

admit no polynomial first integrals.

Indeed, the system (2.12) has Kovalevskaya exponents $\rho_1 = \rho_2 = \rho_3 = -1$. It is easy to verify that conditions (2.4) are not fulfilled for any positive integer M. Moreover, as proved in [2], the system (2.12) has no rational first integrals.

3. Existence of polynomial symmetry fields. Necessary conditions

The first integrals are the simplest tensor invariants of the system (1.1). In [4] Kozlov considered tensor invariants of weight-homogeneous differential equations which include the system (1.1). In particular, he found necessary conditions for the existence of symmetry fields. Below we propose a generalization of his result.

Recall that the linear operator $W = \sum_{i=1}^{n} w_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}$, is called the *symmetry field* of (1.1), if $[W, D_+] = 0$, where D_+ is defined by (2.2). If $w_1, \ldots, w_n$ are homogeneous functions of degree $M + 1$ then the degree of W is M [4].

Theorem 2. *Suppose that the system (1.1) possesses a polynomial symmetry field of degree M and $\rho_1 = -1, \rho_2, \ldots, \rho_n$ are Kovalevskaya exponents. Then there exist non-negative integers $k_2, \ldots, k_n$, $|k| \leqslant M + 1$ such that at least one of the following equalities holds:*

$$k_2\rho_2 + \cdots + k_n\rho_n = M + \rho_i \qquad i = 1, \ldots, n. \tag{3.1}$$

Proof. Let W_M be a polynomial symmetry field of degree M. Substituting in the proof of theorem 2 operators D_+, D_0, D_- with their commutators $[D_+,], [D_0,], [D_-,]$ respectively we repeat the same arguments. □

Corollary 3. *The Halphen equations (2.12) admit no polynomial symmetry fields.*

Using (3.1) we obtain that M may be equal to $-1, 0, 1$ only. It is easy to check that (2.12) does not have symmetry fields of such degrees.

4. Base functions

After the change of variables (2.7) the system (1.1) takes the form

$$\begin{aligned} \dot{y}_1 &= -y_1^2 + \varphi_1(y_2, \ldots, y_n) \\ \dot{y}_i &= (\rho_i - 1)y_1y_i + \varphi_i(y_2, \ldots, y_n) \qquad i = 2, \ldots, n \end{aligned} \tag{4.1}$$

where φ_i are quadratic homogeneous polynomials in the variables $y_2, \ldots, y_n$.

According to (2.2) we now define operators D_+, D_0, D_-.

A homogeneous polynomial $P_M(y_1, \dots, y_n)$ of degree M satisfying the condition

$$D_-(D_+P_M) = 0 \tag{4.2}$$

is called the *base function* of the system (4.1). In other words, the function D_+P_M does not depend on y_1. It is clear that base functions of degree M form a linear space L_M over field $\mathbb{C}$.

Lemma 4. *If the system (4.1) has a homogeneous polynomial first integral F_M of degree M, then $F_M \in L_M$.*

Indeed, by definition, we have $D_+F_M = 0$, hence, in view of (4.2), $F_M \in L_M$.

Let $J(M) = \{z \in \mathbb{Z}_+^{n-1} \mid z_2\rho_2 + \cdots + z_n\rho_n = M, |z| \leqslant M\}$ be the set of integer-valued vectors $z = (z_2, \dots, z_n)^T$ for which the condition (2.11) is fulfilled. We put $m = |J(M)|$ and suppose $J(M) \neq \emptyset$.

Theorem 5. *The dimension d of L_M satisfies the condition* $1 \leqslant d \leqslant m$.

Proof. Let us assume the set $J(M)$ contains vectors $z^{(1)}, \dots, z^{(m)}$ which are ordered by the norm $|z| = z_2 + \cdots + z_n$

$$|z^{(1)}| \leqslant \cdots \leqslant |z^{(m)}|.$$

We define the vector $\rho = (\rho_2, \dots, \rho_n)^T$ and put $(\rho, z) = \rho_2 z_2 + \cdots + \rho_n z_n$, $|z^{(i)}| = n_i$, $i = 1, \dots, m$.

Following the proof of theorem 2, for each $i = 1, \dots, m$ we consider the system of linear partial differential equations

$$\begin{aligned}
&D_0P_{i,n_i} = MP_{i,n_i}\\
&D_0P_{i,n_i+1} = MP_{i,n_i+1} - D_+P_{i,n_i}\\
&\cdots\\
&D_0P_{i,M-1} = MP_{i,M-1} - D_+P_{i,M-2}\\
&D_0P_{i,M} = MP_{i,M} - D_+P_{i,M-1}
\end{aligned} \tag{4.3}$$

$$D_-P_{i,l+1} = (M-l)P_{i,l} \qquad l = n_i, \dots, M-1 \tag{4.4}$$

which defines polynomials $P_{i,n_i}, \dots, P_{i,M}$ recurrently.

It follows from $(z^{(i)}, \rho) = M$ that the first equation in (4.3) has the particular solution $P_{i,n_i} = y_2^{z_2^{(i)}}, \dots, y_n^{z_n^{(i)}}$.

Equations (4.3), (4.4) define a certain base function $P_{i,M}$. Indeed, according to (4.4), we have

$$P_{i,M-1} = D_-P_{i,M}. \tag{4.5}$$

Substituting (4.5) into the last equation in (4.3), and using relations (2.3) we find

$$D_0P_{i,M} = MP_{i,M} - D_+D_-P_{i,M} = MP_{i,M} - (D_-D_+ - D_0 + U)P_{i,M}.$$

Hence $D_-(D_+P_{i,M}) = 0$.

Now consider the problem on the existence of a solution of (4.3), (4.4) in the form of homogeneous polynomials $P_{i,n_i}, \dots, P_{i,M}$. We fix certain $i = 1, \dots, m$ and put $a_i = M - n_i$. Using the relations (4.4) we can write

$$\begin{aligned}
&P_{i,n_i} = I_{i,n_i}\\
&P_{i,n_i+p} = \sum_{j=0}^{p} \binom{p-j}{a_i-j} y_1^{p-j} I_{i,n_i+j} \qquad p = 1, \dots, a_i
\end{aligned} \tag{4.6}$$

where $I_{i,k}(y_2, \dots, y_n)$ are certain homogeneous polynomials of degrees $k = n_i, \dots, M$. Notice that $I_{i,k}$ does not depend on y_1.

Differential operators D_+, D_0 can be represented in the form

$$D_+ = (-y_1^2 + \varphi_1)\frac{\partial}{\partial y_1} + y_1(A_0 - \tilde{U}) + A_+$$
$$D_0 = -y_1\frac{\partial}{\partial y_1} + A_0 \tag{4.7}$$

where

$$A_+ = \sum_{k=2}^{n} \varphi_k \frac{\partial}{\partial y_k} \qquad A_0 = \sum_{k=2}^{n} \rho_k y_k \frac{\partial}{\partial y_k} \qquad \tilde{U} = \sum_{k=2}^{n} y_k \frac{\partial}{\partial y_k}. \tag{4.8}$$

Using (4.3), (4.6), (4.7) one deduces the following equations for determination of I:

$$\begin{aligned}
&A_0 I_{i,n_i} = M I_{i,n_i}\\
&A_0 I_{i,n_i+1} = M I_{i,n_i+1} - A_+ I_{i,n_i}\\
&A_0 I_{i,n_i+2} = M I_{i,n_i+2} - a_i\varphi_1 I_{i,n_i} - A_+ I_{i,n_i+1}\\
&A_0 I_{i,n_i+3} = M I_{i,n_i+3} - (a_i - 1)\varphi_1 I_{i,n_i+1} - A_+ I_{i,n_i+2}\\
&\dots\\
&A_0 I_{i,M} = M I_{i,M} - 2\varphi_1 I_{i,M-2} - A_+ I_{i,M-1}.
\end{aligned} \tag{4.9}$$

We can write each equation of (4.9) as follows:

$$A_0 X_l = M X_l + Y_l \tag{4.10}$$

where X_l, Y_l are homogeneous polynomials of weighted degree $l = n_i, \dots, M$. Let us assume

$$X_l = \sum_{|i|=l} c_{i_2,\dots,i_n} y_2^{i_2}, \dots, y_n^{i_n} \qquad Y_l = \sum_{|i|=l} d_{i_2,\dots,i_n} y_2^{i_2}, \dots, y_n^{i_n} \qquad |i| = i_2 + \dots + i_n \tag{4.11}$$

where $c_{i_2,\dots,i_n}$, $d_{i_2,\dots,i_n}$ are constant parameters. Then substituting (4.11) into (4.10), we obtain the following linear system with respect to $c_{i_2,\dots,i_n}$:

$$(i_2\rho_2 + \dots + i_n\rho_n - M)c_{i_2,\dots,i_n} = d_{i_2,\dots,i_n} \tag{4.12}$$

for $i_2, \dots, i_n = 0, 1, \dots, |i| = l$.

Suppose there exists a set $k_2, \dots, k_n$ for which the following conditions are fulfilled:

$$(k_2, \dots, k_n)^T \in J(M) \qquad d_{k_2,\dots,k_n} \neq 0 \qquad |k| = l. \tag{4.13}$$

Then the solution $I_{i,n_i}, \dots, I_{i,M}$ does not exist. In this case we put $P_{i,M} = 0$.

If the conditions (4.13) are not satisfied, we obtain the base function

$$P_{i,M} = \sum_{j=0}^{a_i} y_1^{a_j - j} I_{i,n_i+j}. \tag{4.14}$$

It is easy to show that polynomials $\{P_{i,M}\}_{i=m}^{i=1}$ are linearly independent over the field $\mathbb{C}$.

Taking into account that $n_1 \leqslant \dots \leqslant n_m$ and using (4.13), we see that in the case $i = m$ we always can determine the base function $P_{i,M}$. Therefore, under the assumption $J(M) \neq \emptyset$, the space L_M always contains a non-zero function. □

Corollary 6. *If at least one resonance condition of the form*

$$(z, \rho) = M \qquad |z| \leqslant M \qquad z \in \mathbb{Z}_+^{n-1}$$

is fulfilled, then there exists a base function of degree M.

5. Polynomial first integrals in the case of a quadratic plane vector field

The first classification of integral curves of two-dimensional quadratic homogeneous systems can be found in the paper by Lyagina [5] and later was completed by numerous authors.

In this section we apply the previous results to this problem to illustrate the method of basis functions.

Consider the system

$$\begin{aligned}\dot{x}_1 &= a_1x_1^2 + b_1x_1x_2 + d_2x_2^2\\ \dot{x}_2 &= a_2x_2^2 + b_2x_1x_2 + d_1x_1^2\end{aligned} \tag{5.1}$$

where a_i, b_i, d_i are constant parameters.

Let $c^{(1)} = (c_1^{(1)}, c_2^{(1)})^T$, $c^{(2)} = (c_1^{(2)}, c_2^{(2)})^T$ be any two linearly independent solutions of the algebraic system (2.1). The exceptional cases when the system (2.1) has only one or admit no solutions are excluded for the discussion below.

We assume that Kovalevskaya exponents corresponding to $c^{(1)}$, $c^{(2)}$ are

$$R_1 = (-1, \rho_1)^T \qquad R_2 = (-1, \rho_2)^T. \tag{5.2}$$

Lemma 7. *The system (5.1) has a homogeneous polynomial first integral of degree M if and only if there exists an integer $k = 1, \ldots, M-1$ such that $\rho_1 = M/k$ and ρ_2 is one of the following numbers:*

$$\frac{M}{M-k}, \frac{M}{M-k-1}, \ldots, \frac{M}{2}, M.$$

Proof. Consider the following change of coordinates:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1^{(1)} & c_1^{(2)} \\ c_2^{(1)} & c_2^{(2)} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{5.3}$$

which exists because of linear independence of vectors $c^{(1)}$, $c^{(2)}$. In coordinates (p_1, p_2) the system (5.1) takes a more simple form

$$\begin{aligned}\dot{p}_1 &= -p_1^2 + (\rho_2 - 1)p_1p_2\\ \dot{p}_2 &= -p_2^2 + (\rho_1 - 1)p_1p_2.\end{aligned} \tag{5.4}$$

It is easy to show that under the change (5.3), the vectors $c^{(1)}$, $c^{(2)}$ turn into $\tilde{c}^{(1)} = (1, 0)^T$, $\tilde{c}^{(2)} = (0, 1)^T$ respectively. Obviously, the system (5.4) has the same Kovalevskaya exponents (5.2). The matrix K, calculated for $\tilde{c}^{(1)}$, is

$$K = \begin{pmatrix} -1 & \rho_2 - 1 \\ 0 & \rho_1 \end{pmatrix}.$$

Under the assumption $\rho_1 \neq -1$, we can reduce K to a diagonal form using the following change of coordinates:

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = L \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

with the constant matrix L

$$L = \begin{pmatrix} 1 & \rho_2 - 1 \\ 0 & \rho_1 - 1 \end{pmatrix}.$$

The case $\rho_1 = -1$ will be considered below.

Finally, equations (5.4) take the form (4.1)

$$\begin{aligned}\dot{y}_1 &= -y_1^2 + \varphi_1\\ \dot{y}_2 &= (\rho_1 - 1)y_1y_2 + \varphi_2\end{aligned} \tag{5.5}$$

where

$$\varphi_1 = ay_2^2 \qquad \varphi_2 = by_2^2$$
$$a = (\rho_2 - 1)(\rho_1 + \rho_2) \qquad b = (\rho_1 - 1)(\rho_2 - 1) - \rho_1 - 1.$$

For the operators (4.8) we get

$$A_+ = \varphi_2 \frac{\partial}{\partial y_2} \qquad A_0 = \rho_1 y_2 \frac{\partial}{\partial y_2} \qquad \tilde{U} = y_2 \frac{\partial}{\partial y_2}.$$

Let F_M be a polynomial first integral of (5.5) of degree M.

According to theorem 2, there exists an integer $k = 1, \ldots, M - 1$ such that

$$k\rho_1 = M. \tag{5.6}$$

We exclude the case $k = M (\rho_1 = 1)$, since if $\rho_1 = \pm 1$, then the system (5.1) has no polynomial first integrals. This can be shown directly using equations (5.4) and (5.5).

Next, we calculate the base function P_M corresponding to the resonance condition (5.6). Consider the equations (4.9). It is obvious that polynomials $I_{1,k}, \ldots, I_{1,M}$ can be represented in the following form:

$$I_{1,k+i} = \alpha_i y_2^{k+i} \qquad i = 0, \ldots, M - k \tag{5.7}$$

where $\alpha_0, \ldots, \alpha_{M-k}$ are constant parameters.

Substituting (5.7) into (4.9) we obtain

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= \frac{bk}{\rho_1(k+1) - M} \\ \alpha_i &= \frac{a(M - k - i + 2)\alpha_{i-1} + b(k + i - 1)\alpha_{i-2}}{\rho_1(k+i) - M} \qquad i = 2, \ldots, M - k. \end{aligned} \tag{5.8}$$

According to (4.14), we get the following expression for the base function P_M:

$$P_M = \sum_{j=0}^{M-k} \alpha_j y_1^{M-k-j} y_2^{k+j}. \tag{5.9}$$

By definition of the base function it is clear that

$$D_+ P_M = \delta y_2^{M+1} \tag{5.10}$$

where

$$\delta = a\alpha_{M-1} + bM\alpha_M. \tag{5.11}$$

Thus, the linear space L_M contains only one polynomial P_M. Hence, taking into account lemma 6, $F_M = \mathrm{const}\, P_M$.

Using (5.10), we conclude that P_M is a first integral if and only if $\delta = 0$. In view of (5.6), (5.8), (5.11) and the above condition, we arrive at lemma 9. □

Theorem 8. *The system (5.1) possesses a homogeneous polynomial first integral of degree M if and only if the following conditions are fulfilled:*

(a) ρ_i, $i = 1, 2$ *are positive rational numbers;*
(b) $\rho_1^{-1} + \rho_2^{-1} \leqslant 1$*;*
(c) $\frac{M}{\rho_i} \in \mathbb{N}$.

This is an obvious consequence of lemma 9.

As an example consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_1^2 - 9x_2^2 \\ \dot{x}_2 &= -3x_1^2 - 8x_1x_2 + 3x_2^2. \end{aligned} \tag{5.12}$$

The vectors $c^{(1)}$, $c^{(2)}$ have the form

$$c^{(1)} = (\tfrac{1}{8}, -\tfrac{1}{8})^T \qquad c^{(2)} = (\tfrac{1}{8}, \tfrac{1}{8})^T.$$

Calculating the corresponding Kovalevskaya exponents (5.2) one obtains

$$R_1 = (-1, 3)^T \qquad R_2 = (-1, \tfrac{3}{2})^T.$$

We have $\rho_1 = 3$, $\rho_2 = \frac{3}{2}$, $\rho_1^{-1} + \rho_2^{-1} = 1$. So, the conditions (a) and (b) of theorem 10 are fulfilled. By the condition (c) one gets $M = 3l$, $l \in \mathbb{N}$. Thus, equations (5.12) possess a cubic first integral F_3. Using formulae (5.8) and (5.9), we obtain

$$F_3 = x_1^3 + x_1^2x_2 - x_1x_2^2 - x_2^3.$$

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